$\mathbb{C}$, the Complex Numbers

Recall a basic (and I mean basic) polynomial $p(x) = x^2 - 1$. The roots, the x such that $p(x) = 0$, of $p(x)$ are ± 1. Recall that for $p(1) = 0$, $p(x)$ must contain at least one $(x - 1)$ such that

$$p(x) = (x - 1)\, p_1(x).$$

We only have the two roots, so we can factor to

$$p(x) = x^2 - 1 = (x - 1)(x + 1).$$

Next, let us consider $p(x) = x^2 + 1$. Since $x^2 \geq 0$ for all $x \in \mathbb{R}$, for all x in the real numbers,

$$x^2 + 1 \geq 1 \quad \text{for all } x \in \mathbb{R},$$

so it has no real roots, none at all. What we need is some value, call it i, such that $i^2 = -1$, leading to

$$p(i) \;=\; (i)^2 + 1 \;=\; (-1) + 1 \;=\; 0.$$

This leads to the full factorization $p(x) = (x - i)(x + i)$.

Well, that problem is solved, in a sense, but i is NOT a real number, it is classified as 'imaginary'. In fact, it is frequently helpful to view all imaginary numbers as, in essence, at 'right angles to reality,' so in a totally separate direction than all real numbers. There is no reason we can't multiply i by real numbers, or why we can't add it to real numbers. Combining real and imaginary numbers creates a 'complex' number, which we will frequently call z. The basic form of is

$$z = a + bi,$$

a is the length in the real direction and b is the length in the imaginary direction. The typical visualization of complex numbers is on a Cartesian plane, with the horizontal axis representing the real numbers and the vertical axis the purely imaginary ones. Everything on this plane taken together forms the complex numbers, written $\mathbb{C}$.

The set $\mathbb{C}$ actually forms what we call a field, which is a short way of saying you can add, subtract, multiply and divide in $\mathbb{C}$. Doing so is actually very easy, usually. What needs to be remembered is that the real (usually a) terms are fundamentally different from the imaginary (b coefficients with i) ones. So, addition is just about keeping them separate:

$$z_1 = a_1 + b_1 i \qquad z_2 = a_2 + b_2 i$$

$$z_1 + z_2 \;=\; (a_1 + b_1 i) + (a_2 + b_2 i) \;=\; (a_1 + a_2) + (b_1 + b_2)i.$$

Subtraction is much the same.

Next, you simply have to recall that $i^2 = -1$. This comes into play with multiplication

$$\begin{aligned} z_1 \times z_2 = (a_1 + b_1 i) \times (a_2 + b_2 i) &= a_1 a_2 + a_1 b_2 i + b_1 i a_2 + b_1 b_2 i^2 \\ &= a_1 a_2 + a_1 b_2 i + b_1 a_2 i - b_1 b_2 \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2)i. \end{aligned}$$

Note that the calculation is not done until the result has been written in the standard $a + bi$ form.

Example

$$\begin{aligned} (2 - 3i) + (1 + i)(5 - 4i) &= (2 - 3i) + (5 - 4i + 5i - 4i^2) \\ &= (2 - 3i) + (5 + i + 4) \\ &= (2 - 3i) + (9 + i) \\ &= 11 - 2i \end{aligned}$$

Absolute values are slightly different with complex numbers. We define

$$|z| = |a + bi| = \sqrt{a^2 + b^2}.$$

Here it helps to view imaginary numbers as at right angles to real numbers. If that is the case, then z is the hypotenuse of a right angled triangle with sides a and bi, with lengths $|a|$ and $|b|$. Applying Pythagoras' Theorem gives the length of z.

There is one unique operation applied to complex numbers called the complex conjugate. The conjugate of z is written $\overline{z}$, and

$$\overline{z} = \overline{a+bi} = a - bi.$$

It simply inverts the imaginary component. That's it. The conjugate has several uses. First,

$$z\,\overline{z} = (a+bi)(a-bi) = a^2 - abi + abi - b^2i^2 = a^2b^2 = |z|^2.$$

Finally we get to division. We introduced the conjugate first since we will actually make use of it. We will use the old mathematical trick: multiply by one. The particular one we will use is the conjugate of z_2, $\overline{z_2}$, divided by itself.

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{(\overline{z_2})}{(\overline{z_2})} = \frac{z_1\overline{z_2}}{|z_2|^2}$$

and now we need to switch to the a and b format.

$$\begin{aligned}
\frac{z_2}{z_2} = \frac{z_1\overline{z_2}}{|z_2|^2} &= \frac{(a_1+b_1i)(a_2-b_2i)}{a_2^2+b_2^2} \\
&= \frac{a_1a_2 - a_1b_2i + a_2b_1i - b_1b_2i^2}{a_2^2+b_2^2} \\
&= \frac{a_1a_2 + b_1b_2 + (-a_1b_2 + a_2b_1)\,i}{a_2^2+b_2^2} \\
&= \frac{a_1a_2+b_1b_2}{a_2^2+b_2^2} + \left(\frac{-a_1b_2+a_2b_1}{a_2^2+b_2^2}\right)i.
\end{aligned}$$

Again, the question isn't answered unless it has been written in the $a+bi$ format.

Example

$$\begin{aligned}
\frac{3-4i}{1+i} &= \frac{(3-4i)}{(1+i)}\frac{(1-i)}{(1-i)} \\
&= \frac{3-i-4i+4i^2}{1-i+i-i^2} \\
&= \frac{3-5i-4}{1+1} \\
&= \frac{-1-5i}{2} \quad = \frac{-1}{2} - \frac{5}{2}i
\end{aligned}$$

One more thing. Remember the polynomials and their factors and roots? Well, $\mathbb{C}$ allows us to fully factor any polynomial.

The Fundamental Theorem of Algebra: All non-constant polynomials have a root in $\mathbb{C}$.

The theorem actually covers more than the standard real valued polynomials. It actually covers polynomials with complex coefficients.

The fundamental theorem may not seem like much, but we can easily expand it.

Corollary: Any nth order polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

can be factored into the form

$$p(x) = a_n(x - z_1)(x - z_2)\cdots(x - z_{n-1})(x - z_n).$$

This is easy to prove. You start with the $p(x)$. If it is constant, then its order zero and we simply get $p(x) = a_0$. Otherwise, we get our first root in $\mathbb{C}$, z_1. We can factor out $(x - z_1)$ for

$$p(x) = (x - z_1)p_1(x).$$

We can do that again, and again, until our, eventual, $p_n(x)$ is simply a constant, a_n.

One thing to note: the z_1 to z_n need not be unique, we may have repeats.

Example

If we want to factor a quadratic $x^2 - 6x + 10$ then we need only use the quadratic formula:

$$ax^2 + bx + c \qquad \text{roots are} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which, in this case, is

$$x = \frac{6 \pm \sqrt{36 - 40}}{2} = \frac{6 \pm \sqrt{-4}}{2} = \frac{6 \pm (2i)}{2} = 3 \pm i.$$

This means that the factors are $3+i$ and $3-i$, or, rather, $3+i$ and $\overline{3-i}$. The complex conjugate is built into the roots of real quadratics. Turns out, it is built into ALL real polynomials.

Useful Property: Take a real polynomial $p(x)$, a polynomial with real coefficients. If $p(x)$ has a root z then $p(x)$ must also have the root $\overline{z}$ (recall the complex conjugate). Proving this is difficult, but it is fairly easy to see why this should be true, see the previous example. In general, see what happens if we multiply $(x-z)$ with $(x-\overline{z})$

$$\begin{aligned}\big(x-(a+bi)\big)\big(x-(a-bi)\big) &= x^2 - x(a-bi) - x(a+bi) - (a+bi)(a-bi)\\ &= x^2 - x(a-bi+a+bi) - \big(a^2 - abi + abi - b^2i^2\big)\\ &= x^2 - 2ax - (a^2+b^2).\end{aligned}$$

Including the conjugate as another root clears out the imaginary components. All roots of real polynomials come in conjugate pairs. Note, that this applies meaningfully only to strictly complex roots. Real numbers are unchanged by complex conjugation, so this applies very trivially to real roots (i.e., if $k \in \mathbb{R}$ is a root of $p(x)$ then $\overline{k} = k$ is a root of $p(x)$).

Example: we will now factor the polynomial $p(x) = x^3 - 5x^2 + 9x - 5$ knowing that $2+i$ is a root.

First, since we have $2+i$ we will also get $2-i$. That means that

$$(x-2-i)(x-2+i) = x^2 - 4x + 5$$

is a factor of $p(x)$. So, we divide that out of $p(x)$, leading to

$$x^3 - 5x^2 + 9x - 5 = (x-1)(x^2-4x+5) = (x-1)(x-2-i)(x-2+i).$$

Exercises From the Text

Complex numbers are covered in the text in section 2.5, subsections 1, 2 and 4. Here are a few relevant questions from the section, which should have answers in the back of the book:

1.bdfhj) 2.b) 3.d) 4. 6.b) 8.d) 9.d)

A note about 9d): By 'irreducible' they mean a quadratic that cannot be factored in $\mathbb{R}$.